

# Holographic Non-Fermi Liquids and the Luttinger Theorem

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## Abstract

We show that the Luttinger theorem, a robust feature of Fermi liquids, can be violated in non-Fermi liquids. We compute non-Fermi liquid Green functions using duality to black holes and find that the volume of the Fermi surface depends exponentially on the scaling dimension, which is a measure of the coupling. This demonstrates that Luttinger's theorem does not extend to non-Fermi liquids. We comment on possible experimental signatures.

## I. INTRODUCTION

Though many metals are well described by Landau Fermi-liquid theory, there are numerous examples of materials that are not well-described by weakly interacting quasiparticles [1]. Non-Fermi liquids, which include the normal phase of high- $T_c$  superconductors, are inherently strongly coupled, which makes them interesting systems to study. Luttinger's theorem [2], the constancy of the phase-space volume contained within the Fermi surface as a function of coupling, is a robust feature of Fermi liquids. It is not *a priori* obvious, however, that this theorem persists in non-Fermi liquid systems, even when a notion of a Fermi surface does apply. In this Letter we show that Luttinger's theorem is violated in a particular non-Fermi liquid system.

Since non-Fermi liquids are inherently strongly coupled, finding tractable fora for investigating them is difficult. Beginning with [3] there has been considerable interest in studying strongly coupled fermions using gauge/gravity duality [4]. Although there have been suggestions of the formation of Fermi surfaces in string-theoretic brane constructions [5, 6], the bottom up methods for studying probe fermions in charged black hole backgrounds developed in [3, 7–9] provide the most convenient setting for studying non-Fermi liquid behavior.

In this Letter we use holographic techniques to show that Luttinger's theorem can be violated in non-Fermi liquids. We study fermionic correlation functions in anti-de Sitter (AdS) black hole backgrounds using probe fermions of various masses. Via the holographic dictionary for fermions [10] the conformal dimension of the dual operator in the boundary conformal field theory (CFT) is controlled by the mass of the bulk fermion. In [8] it was found that, if the mass was tuned so that the conformal dimension of the operator in the boundary theory coincided with that of a free fermion, the spectral function exhibited a peak consistent with that of a Fermi liquid. Tuning the mass so that the conformal dimension departed from the free value yielded behavior that deviated from the Fermi liquid. The authors of [8] interpreted the mass, then, as a proxy for the coupling, and found that the Fermi momentum remained constant as the coupling was varied, as predicted by Luttinger's theorem.

However, the fermionic spectral function in the background of a charged black hole has multiple peaks, and the peak under consideration in [8] is different than the non-Fermi liquid peak considered in [7]. In this Letter we study the spectral peak in [7] as a function of the

fermion mass. Following [8], we interpret the mass as a proxy for the coupling. We find that the Fermi momentum depends exponentially on the probe fermion mass, and thereby on the coupling, in clear violation of Luttinger's theorem.

We consider boundary CFTs that are both 2 + 1- and 3 + 1-dimensional. In the former case, the result persists in the presence a magnetic field [11]. In that case, the bulk wave equation can be expanded in terms of Landau levels, and the resulting wave equation can be reduced to the electrically charged case [12, 13]. As we will discuss, this allows for a possible experimental signature of this effect in de Haas-van Alphen measurements.

## II. SETUP

We start with the Reissner-Nordström-AdS black hole in  $d+1$  dimensions with the metric

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + \frac{r^2}{L^2}dx_i^2, \quad (1)$$

with (e.g. [14])

$$f(r) = \frac{r^2}{L^2} - \frac{M}{r^{d-2}} + \frac{Q^2}{r^{2(d-2)}},$$

$$A = \left( \mu - \sqrt{\frac{d-1}{2(d-2)}} \frac{Q}{r^{d-2}} \right) dt. \quad (2)$$

We introduce an orthonormal frame according to

$$e^0 = \sqrt{f(r)}dt, \quad e^i = \frac{r}{L}dx^i, \quad e^d = \frac{dr}{\sqrt{f(r)}}, \quad (3)$$

where  $i = 1, \dots, (d-1)$ . The spin connection is

$$\omega^{ab} = \frac{f'(r)}{2}(\delta_0^a \delta_d^b - \delta_d^a \delta_0^b)dt + \frac{1}{L}\sqrt{f(r)}(\delta_i^a \delta_d^b - \delta_d^a \delta_i^b)dx_i. \quad (4)$$

We will scale  $Q$ ,  $M$  by a dimensionful parameter  $r_h$  as

$$Q \rightarrow \frac{r_h^{d-1}}{L}Q, \quad M \rightarrow \frac{r_h^d}{L^2}M. \quad (5)$$

If the dimensionless charges are related by  $M = 1 + Q^2$  then  $r_h$  is the largest zero of  $f(r)$  and therefore the horizon radius. We also adjust  $\mu$  so that the gauge field vanishes at the horizon

$$A = \mu \left( 1 - \frac{r_h^{d-2}}{r^{d-2}} \right) dt, \quad \mu = \sqrt{\frac{d-1}{2(d-2)}} Q \frac{r_h}{L}. \quad (6)$$

The temperature of the black hole is

$$T = \frac{r_h}{4\pi L^2}(d - (d-2)Q^2), \quad (7)$$

whereby the extremal black hole has  $Q^2 = \frac{d}{d-2}$ .

The action for a minimally coupled fermion is

$$S_{\text{Dirac}} = \int d^{d+1}x \sqrt{-g} i(\bar{\psi} \Gamma^a e_a^\mu D_\mu \psi - m \bar{\psi} \psi), \quad (8)$$

where

$$\bar{\psi} = \psi^\dagger \Gamma^0 \quad D_\mu = \partial_\mu + \frac{1}{8} \omega_\mu^{ab} [\Gamma_a, \Gamma_b] - iq A_\mu. \quad (9)$$

It is convenient to work in terms of eigenspinors  $\psi_\pm = \Gamma_\pm \psi$  of the projection operators  $\Gamma_\pm = \frac{1}{2}(1 \pm \Gamma^d)$ . We will also consider  $m > 0$  without loss of generality.

Defining now

$$\psi_\pm = (f(r))^{-\frac{1}{4}} r^{-\frac{d-1}{2}} e^{-i\omega t + ik_i x^i} \phi_\pm, \quad (10)$$

the Dirac equation has the form

$$r\sqrt{f(r)} \left( \partial_r \mp \frac{m}{\sqrt{f(r)}} \right) \phi_\pm = \mp i \gamma^\mu K_\mu L \phi_\mp, \quad (11)$$

where

$$K^\mu = (u, k_i), \quad u = \frac{r}{L\sqrt{f(r)}}(\omega + qA_t), \quad (12)$$

and  $\gamma^\mu$  are  $d$ -spacetime-dimensional gamma matrices.

If  $mL < \frac{1}{2}$ , then near the boundary

$$\begin{aligned} \phi_+ &\approx A \left( \frac{r}{r_h} \right)^{mL} + B \left( \frac{r}{r_h} \right)^{-mL-1}, \\ \phi_- &\approx D \left( \frac{r}{r_h} \right)^{-mL} + C \left( \frac{r}{r_h} \right)^{mL-1}, \end{aligned} \quad (13)$$

where the coefficients are related by

$$C = \frac{iL^2 \gamma^\mu k_\mu}{(2mL-1)r_h} A, \quad B = \frac{iL^2 \gamma^\mu k_\mu}{(2mL+1)r_h} D, \quad (14)$$

and  $k^\mu = (\omega + q\mu, k_i)$ .

Since we have rotational symmetry on the boundary, pick  $\vec{k} = (k, 0, \dots, 0)$ , and the gamma matrices so that  $\gamma^0 = i\sigma_2$ ,  $\gamma^1 = \sigma_1$  and  $\phi_\pm = \begin{pmatrix} y_\pm \\ z_\pm \end{pmatrix}$ . The Dirac equation can then

be written as

$$\begin{aligned} r\sqrt{f(r)}\left(\partial_r \mp \frac{m}{\sqrt{f(r)}}\right)y_{\pm} &= \mp iL(k-u)z_{\mp}, \\ r\sqrt{f(r)}\left(\partial_r \mp \frac{m}{\sqrt{f(r)}}\right)z_{\pm} &= \mp iL(k+u)y_{\mp}. \end{aligned} \quad (15)$$

Let us henceforth consider the extremal black hole. The leading order equation of motion for  $y_+$  in the near horizon region  $r = r_h + L\epsilon$  for  $\epsilon \ll r_h/L$  can be written as

$$(\epsilon^2\partial_\epsilon + \tilde{m}\epsilon)(\epsilon^2\partial_\epsilon - \tilde{m}\epsilon)y_+ = -\tilde{\omega}^2y_+, \quad (16)$$

where  $\tilde{m} = mL/\sqrt{d(d-1)}$  and  $\tilde{\omega} = \omega L/d(d-1)$ . The corresponding equation for  $z_+$  is identical. This equation can be solved exactly in terms of Hankel functions, however we need only the leading small  $\epsilon$  behavior. Selecting the solution that is purely ingoing at the horizon, appropriate for computing retarded correlation functions, we can choose overall normalizations so that

$$y_+ \approx z_+ \approx y_- \approx -z_- \approx e^{i\frac{\tilde{\omega}}{\epsilon}}. \quad (17)$$

The response function (for  $mL > 0$ ) is computed as

$$G_R = \lim_{\epsilon \rightarrow 0} \epsilon^{-2mL} \left( \begin{array}{cc} \xi_+ & 0 \\ 0 & \xi_- \end{array} \right) \bigg|_{r=\frac{r_h}{\epsilon}}, \quad (18)$$

where

$$\xi_{\pm} = \left( i \frac{y_{\mp}}{z_{\pm}} \right)^{\pm 1}, \quad (19)$$

satisfies the equation of motion

$$r\sqrt{f(r)}\partial_r\xi_{\pm} = -2rm\xi_{\pm} \mp L(k \mp u) \pm L(k \pm u)\xi_{\pm}^2. \quad (20)$$

Ingoing boundary conditions at the horizon take the form

$$\xi_{\pm}|_{r=r_h} = i. \quad (21)$$

We will compute the Green function (18) by solving the equation of motion (20) numerically. The ratio (19) is convenient for this because, as a ratio of waves oscillating in phase, it is not itself oscillatory. Also, since  $f(r_h) = 0$ , for numerical purposes we expand the equation of motion near  $r = r_h$  and begin the numerical evolution just outside the horizon.

### III. LUTTINGER THEOREM

The Luttinger theorem [2] states that the volume of phase space contained within the Fermi surface remains constant as the coupling between quasiparticles is adjusted. In [2], the Fermi surface was defined as the location of a discontinuity in the density of states. Standard Landau Fermi liquid theory [2, 15] associates this discontinuity with a peak in the fermion Green function. In isotropic situations, such as the one we are considering, the Fermi surface is a sphere, which means that constancy of the volume enclosed by the Fermi surface implies constancy of the Fermi momentum. If holographic non-Fermi liquids obey the Luttinger theorem, the non-Fermi liquid peak would remain at a fixed momentum as the coupling is adjusted. We will see that this is not the case.

Holography associates the scaling dimension of CFT operators with the asymptotic scaling behavior of the bulk gravity field. Generically, this scaling behavior is determined by the dimensionality of the bulk, the spin of the bulk field, and its mass. Since the scaling dimensions of operators flow with the coupling strength, adjusting the mass of the bulk field, and thereby its scaling dimension, can be interpreted as adjusting the coupling in the CFT.

Let us concentrate for now on the situation in  $AdS_4$ -Reissner-Nordström. The Green function in this background has been found to exhibit two peaks. One is the peak in  $\text{Im}(G_{--})$  that was identified in [7] and is associated with non-Fermi liquid behavior. The other found in [8], is in  $\text{Im}(G_{++})$  [16] in this notation, and is associated with Fermi-liquid behavior. The Fermi-liquid peak studied in [8] was found to remain at approximately fixed momentum as a function of the bulk mass, in agreement with Luttinger's theorem. We find that this does not apply for the non-Fermi liquid peak.

In [7] the Fermi momentum for zero fermion mass was determined to be  $k_F(0) \simeq 0.918528499(1) r_h/L^2$ . In [17] it was shown that this value can be found as the solution to an equation involving hypergeometric functions. We are interested in determining the Fermi momentum for other values of the mass parameter. In figure 1 we plot  $\text{Im}(G_{--})$  for  $mL = 0.3$  as a function of the frequency for two different values of the momentum. As  $k$  approaches  $k_F$  from below, the peak in the top plot in figure 1 becomes sharper and approaches  $\omega = 0$ . Above  $k_F$  the height of the peak is reduced to two bumps as seen in the bottom plot in figure 1. In [7] the Fermi momentum was determined by following the location of the peak to  $\omega = 0$ . This procedure is somewhat laborious, so we used a numerical

routine to automate its determination, at the expense of reduced precision. We solve the equation of motion (20) for small but non-zero  $\omega$  and  $k < k_F$ . We do this repeatedly for  $k < k_F$ , for each  $k$  determining the frequency  $\omega^*$  where  $\text{Im}(G_{--})$  is maximum. We then suppose that near  $k_F$

$$\text{Im}(G_{--}(\omega^*, k)) \approx \frac{c}{(k_F - k)^\alpha}, \quad (22)$$

to obtain an estimate of  $k_F$ .

In figure 2 we plot the ratio of the Fermi momentum at a given mass to the Fermi momentum at zero mass as determined in [7]. Our numerical routine yielded results that fell into two distinct groupings that differed by a few orders of magnitude in the goodness of fit to the extrapolation of the pole (22). We kept only those points in the group with the better fit. Among this grouping it also yielded two groups of values of  $k_F$  that were determined to about  $10^{-2}$  and those determined to about  $10^{-4}$ . We kept only those values of  $k_F$  that were determined to better than  $10^{-4}$ . We have plotted the ratio  $k_F/k_F(0)$  on a logarithmic scale to emphasize that the Fermi momentum appears to fall off exponentially from the value it takes at zero fermion mass. We determined the rate of fall-off to be

$$k_F \approx k_F(0) \exp(-\sigma mL), \quad \sigma \simeq 1.148 \pm 0.002. \quad (23)$$

The change in the Fermi momentum occurs at fixed chemical potential, and fixed charge density in the boundary theory, as they are determined by the background gauge field. As such the change in the Fermi momentum cannot be ascribed to a change in the density of charge carriers. The fact that this ratio is not unity for all values of the mass indicates a violation of Luttinger's theorem.

The scaling form (23) suggests that a novel scaling law replaces Luttinger's theorem. The bulk mass is related to the conformal dimension of the boundary field as  $\Delta = \frac{d}{2} + |mL|$ , so the scaling relation can be recast as  $k_F \propto e^{-\sigma\Delta}$ . This form does not refer to the holographic set-up and could be valid for non-Fermi liquids more generally. The parameter  $\sigma$  should measure violations of the Luttinger theorem in more general settings as well, such as the semi-holographic Fermi-liquid [18].

We repeated the analysis for the  $AdS_5$ -Reissner-Nordström black hole, corresponding to  $3 + 1$  dimensional condensed matter systems. We again found a violation of Luttinger's theorem of the scaling form (23), now with  $k_F(0) \simeq 0.8155$  and  $\sigma \simeq 0.80$ .

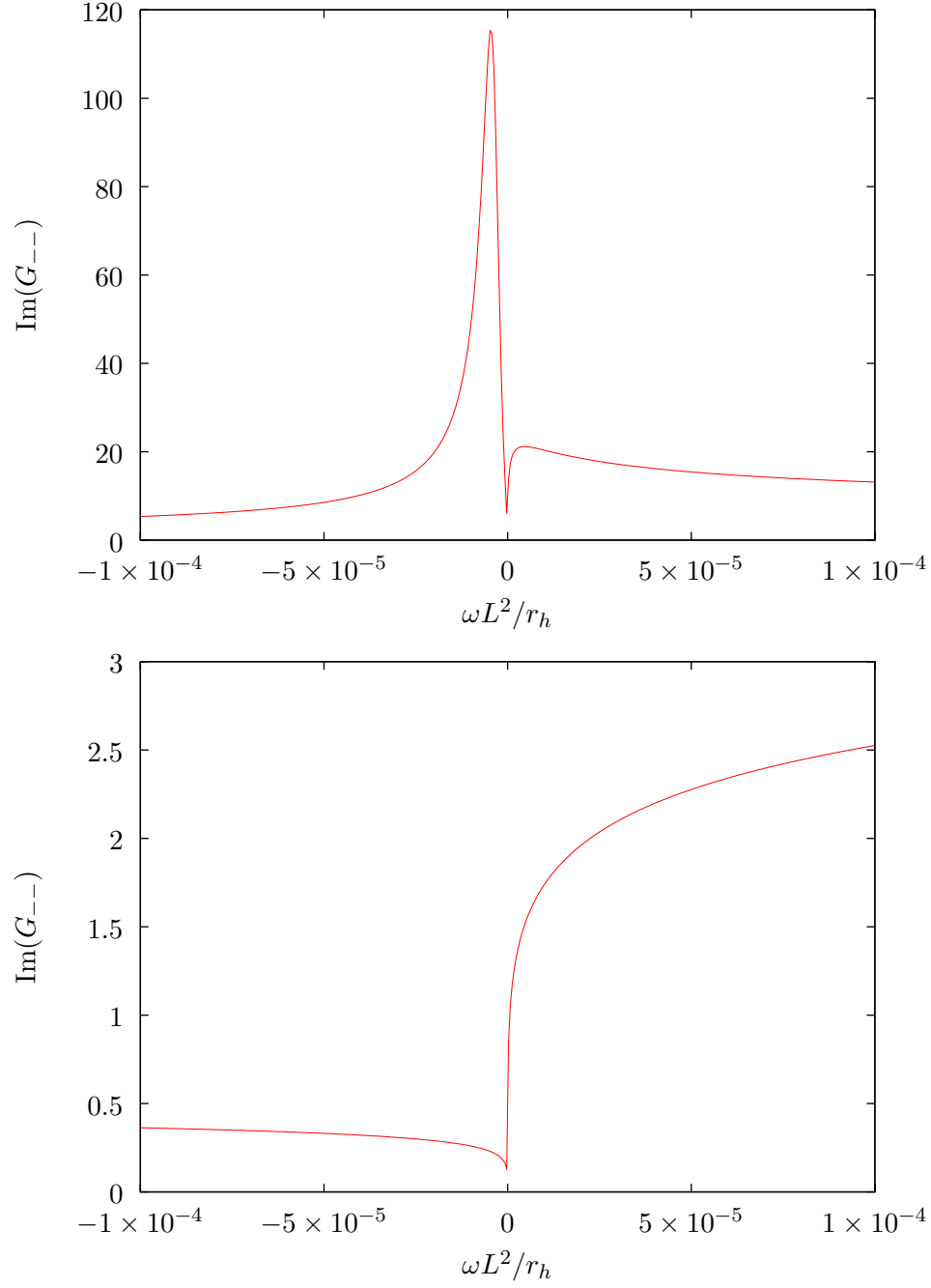


FIG. 1. Plot of  $\text{Im}(G_{--})$  as a function of  $\omega$  for  $mL = 0.3$  for extremal  $AdS_4$ -RN. The top plot is at fixed  $k = 0.6 r_h / L^2$  which is less than our estimated value of  $k_F \simeq (0.65308 \pm 0.00009) r_h / L^2$  for this mass and the bottom plot is for  $k = 0.7 r_h / L^2$ .



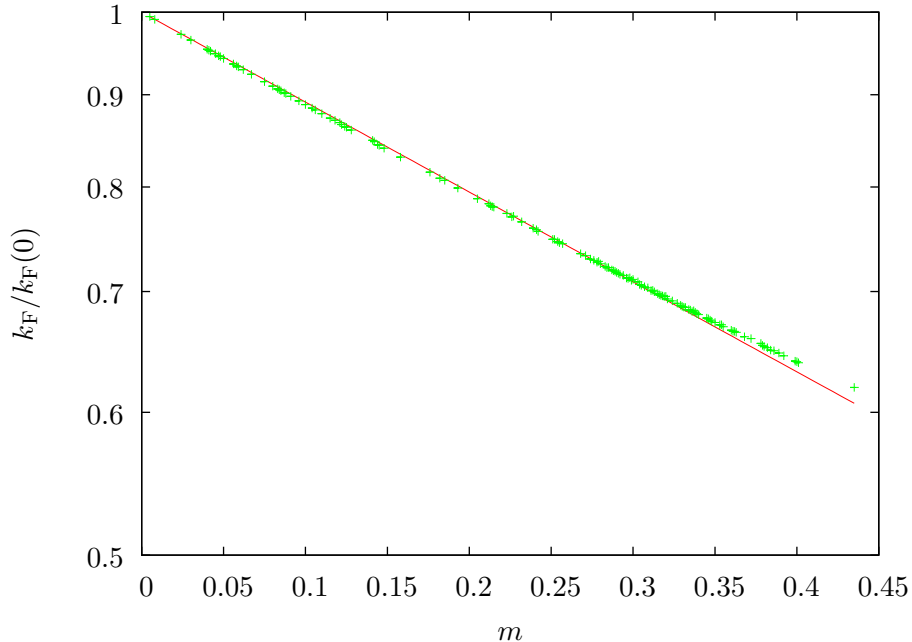


FIG. 2. Plot of  $k_F/k_F(0)$  vs  $m$  for extremal  $AdS_4$ -RN. The line is a fit to the data in the form of (23).

#### IV. DISCUSSION

We have shown that a non-Fermi liquid may violate Luttinger's theorem in a specific manner. It is interesting to consider how this effect could be detected in real condensed matter systems. One characteristic manifestation of the Fermi surface is de Haas-van Alphen effect, the oscillation of the magnetic susceptibility as a function of the inverse magnetic field with period

$$\delta\left(\frac{1}{B}\right) \propto \frac{1}{A_{\text{FS}}}, \quad (24)$$

where  $A_{\text{FS}}$  is the area of the Fermi surface. A  $B$ -field is readily incorporated by a dyonic black hole (considered in [12, 13, 19]) and much of the computation we performed carries over, with the continuous momentum replaced by Landau levels. In particular, (20) remains the equation of motion after appropriately replacing  $k$ . We then expect that the exponential relation (23) between Fermi momentum and mass will persist in the presence of a magnetic field.

For zero mass it is known that adjustment of the magnetic field leads to oscillations in the semi-classical theory that can be interpreted as the de Haas-van Alphen effect [12, 13].

In view of our result, we expect the change in the period of de Haas-van Alphen oscillations (24) to be

$$\delta \left( \frac{1}{B} \right) \propto e^{2\Delta\sigma}, \quad (25)$$

where  $\Delta$  is the scaling dimension of the field, and  $\sigma$  is the scaling constant in (23). The scaling dimension for various currents can be determined at vanishing  $B$ -field and then  $\sigma$  can be measured from (25), at least in principle.

One might also hope to uncover the behavior we have found here using photo-electric scattering techniques [20]. These techniques directly determine the low energy density of states, and are therefore effective at measuring the Fermi surface. In systems that are well-described by the holographic non-Fermi liquid we have studied here, we would expect the size of the Fermi surface to scale exponentially with the conformal dimension of the operator with the scaling rate given in (23).

It is an interesting open question whether the behavior we have seen here occurs in other holographic settings. Of particular interest is the background considered in [21]. It was shown there that the low temperature phase of Einstein-Maxwell-Chern-Simons gravity is dominated by a solution with vanishing entropy, rather than  $AdS_5$ -Reissner-Nordström. Since that solution satisfies the third law it constitutes an interesting setting for an examination along the lines of what we have done here.

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